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THE NON-LINEAR ACTION OF TANGENTIAL STRESSES ON THE WAVE MOTION OF A LOW-VISCOSITY FLUID*

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Formal asymptotic expansions of the solution of a non-linear problem on the wave motion of a fluid with specified tangential surface stresses are constructed at high Reynolds numbers. A non-linear boundary layer (BL), for which a selfsimilar solution is constructed, is formed close to the free boundary. The flow outside of the BL satisfies Euler's equation. The free boundary is determined by a dynamic condition which takes account of the tangential stresses and the velocity field in the BL. The action of the tangential stresses on solitary waves and on low amplitude progressive waves is calculated numerically.

Non-linear BL's close to free boundaries when there is thermocapillary flow have been studied in /1-4/. The action of tangential stresses on the wave motions of a fluid in the case of a disappearing viscosity has been treated in a linear formulation in /5, 6/. Asymptotic expansions of the solution of a stationary non-linear problem with a free boundary have been constructed in /7, 8/.

1. A non-linear problem is considered concerning the wave motion of a fluid under the action of a system of "travelling" tangential stresses $T(x-ct)$, specified on a free boundary Γ, for a system of Navier-Stokes equations with a disappearing viscosity $\nu\to 0$

$$\begin{gathered}\partial \mathbf{v}/\partial t+(\mathbf{v},\nabla)\mathbf{v}=-\rho^{-1}\nabla p+\nu\Delta\mathbf{v}+\mathbf{g},\quad \operatorname{div}\mathbf{v}=0\\ p=2\nu\rho\mathbf{n}\Pi\mathbf{n}+\sigma k+p_*,\quad 2\nu\rho\Pi\mathbf{n}-2\nu\rho(\mathbf{n}\Pi\mathbf{n})\mathbf{n}=T(x-ct),\\ \partial G/\partial t+\mathbf{v}\nabla G=0,\quad (x,z)\in\Gamma\end{gathered}\tag{1.1}$$

Here $\mathbf{v}=(v_x,v_z)$, $\mathbf{g}=-g\mathbf{e}_z$, $\mathbf{e}_z$ is a unit vector along the vertical z-axis, g is the gravitational acceleration constant, ρ is the density, k is the curvature of the free boundary Γ (it is assumed that $k>0$, if the boundary Γ is convex), σ is the surface tension, $\mathbf{n}$ is the unit vector of the external normal to the free boundary, Π is the rate of deformation tensor, $p_*=\text{const}$ and T are the specified pressure and tangential stress on the free boundary, c is the rate of displacement of the tangential load and $G(x,z,t)=0$ is

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the equation of the free boundary in implicit form. It is assumed that the fluid occupies a horizontal layer D bounded from above by the free boundary Γ and from by a wall S on which the sticking condition is satisfied. The velocity field at infinity is specified. The function $T(x-ct)$ is assumed to be integrable on the surface Γ. Initial conditions are not specified since the solution of the problem is constructed in the form of travelling waves of the form $\mathbf{v}(x-ct)$.

A boundary-layer is formed close to the free boundary and the solid wall. In the unbounded domain everywhere outside of the boundary-layer, the flow is approximately described by Euler's equations. Formal asymptotic expansions of the solution as $\nu\to 0$ of problem (1.1) are constructed below in the case when the characteristic value of the velocity U in the BL close to the free boundary is comparable as regards its order of magnitude ($U\sim c$) or far greater than the rate of propagation of the tangential stresses. In this case a non-linear boundary-layer arises close to the free boundary. If, however, $U\ll c$, the boundary layer equations are linearized and solved in quadratures /8/.

Problem (1.1) is reduced to a dimensionless form and a small parameter $\varepsilon=(\rho\nu^2L^{-2}T_*^{-1})^{1/3}$ is introduced where L and T_* are the characteristic scales of length and of the tangential stress. We note that small values of the coefficient of viscosity correspond to small ε. The characteristic value of the velocity in the boundary-layer close to the free boundary $U=(LT_*^2\nu^{-1}\rho^{-2})^{1/3}$ is adopted as the velocity scale. A dimensionless pressure is introduced by the relationship $p=T_*p'-\rho gz$. Asymptotic expansions of the solution of problem (1.1) when $\varepsilon\to 0$ are constructed in the form

$$\mathbf{v}\sim\mathbf{h}_0+\varepsilon^{1/3}(\mathbf{v}_1+\mathbf{h}_1+\mathbf{w}_1)+\ldots \tag{1.2}$$
$$p'\sim q_0+p_0+\varepsilon^{1/3}(p_1+q_1+r_1)+\ldots,\quad \zeta\sim\zeta_0+\varepsilon^{1/3}\zeta_1+\ldots$$

Here, $z=\zeta(x-ct)$ is the equation of the free boundary.

We denote by D_Γ and D_S the domains of the boundary layers close to the free boundary and the solid wall respectively. Then, $\mathbf{h}_n$ and q_n are functions of the type of solutions of the boundary-layer problem in D_Γ and $\mathbf{w}_1$ and r_1 are functions of the type of the solutions of the boundary-layer problem in the domain D_S. These functions and their derivatives disappear outside of the boundary layers. The functions $\mathbf{v}_1$, p_0 and p_1 determine the solution of the problem outside of the domains D_Γ and D_S.

We note that the characteristic velocity scale U, the orders of the principal terms in the expansions (1.1) and the orders of the thicknesses of the boundary layers are found from the condition for the orders of the viscous and inertial terms in the system of Navier-Stokes equations and in the boundary conditions for the tangential stresses to be the same. In this case the thickness of the boundary-layer close to the free boundary is of the order of ε.

Remark. Asymptotic expansions of the solution of problem (1.1) have been constructed in a linear formation in /5/. Both in the non-linear problem and in the linearized problem /5, 6/, in the case of finite tangential stresses the velocity of the fluid in the boundary-layer close to the free boundary is greater by an order of magnitude than the velocity of the external flow (expansions (1.2)). Taking the non-linearity into account changes the thickness of the boundary layer by δ_n: in the linear problem $\delta_n\sim\nu^{1/2}$ and in the non-linear problem $\delta_n\sim\nu^{2/3}$.

2. The boundary-value problem for the principal terms of the asymptotic forms (1.2) which determine the flow in the boundary-layer close to the free boundary is obtained by the application of a second iterative process to system (1.1) using the Vishik-Lyusternik method /9/.

We now introduce a new coordinate system $x_1=x-ct$, $z_1=z$, which moves along the x-axis at a velocity c together with the system of tangential stresses. Close to the free surface in the moving coordinates, we also introduce the local orthogonal coordinates ξ,φ /8/, where ξ is the distance of the point $N(x_1,z_1)$ from the surface Γ and φ is the curvilinear coordinate, on the surface Γ, of the base of the normal dropping down from the point N. It is assumed that, for sufficiently small ξ, segments of the normals to the surface Γ do not intersect.

Let $h_{\varphi n}$ and $h_{\xi n}$ be the components of the vector h_n in the local coordinates. Now let us determine the velocity field which is independent of t in the travelling coordinate system $h_n(\varphi,\xi)$. We substitute (1.2) into (1.1) and expand $\mathbf{v}_1$, p_0 and p_1 in Taylor series in powers of ξ and put $\xi=\varepsilon s$. By equating the coefficients accompanying ε^{-1} and ε^0 to zero, we find that $h_{\xi 0}=0$ and, in the case of $h_{\varphi 0}$ and $h_{\xi 1}$, we derive a boundary-value problem taking the coordinate φ as the length of an arc along the free boundary

$$(h_{\varphi0}+V)\frac{\partial h_{\varphi0}}{\partial\varphi}+H_{\xi2}\frac{\partial h_{\varphi0}}{\partial s}=\frac{\partial^2 h_{\varphi0}}{\partial s^2},\quad \frac{\partial h_{\varphi0}}{\partial\varphi}+\frac{\partial H_{\xi2}}{\partial s}=0 \qquad (2.1)$$
$$\frac{\partial h_{\varphi0}}{\partial s}=-T(\varphi),\quad H_{\xi2}=0\quad (s=0)$$
$$h_{\varphi0}=0\,(s=\infty);\quad H_{\xi2}=h_{\xi2}+\nu_2\mathbf{n}\,|_\Gamma,\quad V=c/U$$

The vector-function $\mathbf{h}_1$ satisfies a linear boundary-value problem which is not given here.

We note that the boundary-value (2.1) when $V=0$ has been formulated /1, 2/ in the case of Marangoni boundary layers in the investigation of the thermocapillary effect close to a free boundary. The conditions for the solvability of (2.1) in a finite interval when the velocity profile in the initial cross-section in D_Γ is specified have been found in /3/.

Let us now carry out the selfsimilar solution of problem (2.1) when $T=\tau/\sqrt{\varphi}$. We note that, when $V=0$, the selfsimilar solutions have been found in /1, 2/ in the neighbourhood of the critical point.

We introduce the stream function ψ by the relationships $h_{\varphi0}=\partial\psi/\partial s$, $H_{\xi2}=-\partial\psi/\partial\varphi$ and put $\psi=\sqrt{\varphi}\tau^{1/3}F(\eta)$, where $\eta=s\tau^{-1/3}/\sqrt{\varphi}$. We obtain a boundary-value problem for the function $F(\eta)$ (it is not necessary to specify the initial profile since it is determined by the selfsimilarity condition).

$$2F'''+FF''-\beta\eta F''=0,\quad \beta=V\tau^{-2/3} \qquad (2.2)$$
$$F''(0)=-1,\quad F(0)=F'(\infty)=0$$

The solution of problem (2.2) is constructed numerically using the Runge-Kutta method for different $\beta<0$. A plot of the function $F'(0)$ (the velocity of the fluid on the free boundary) is shown by the solid line in Fig.1, while a plot of $F(\infty)$ as a function of the parameter $|\beta|$ is shown by the broken line. As $|\beta|$ increases, the magnitude of $F'(0)$ decays monotonically from a value of 1.7188 when $\beta=0$ to zero when $|\beta|\to\infty$. when $|\beta|\gg1$, the asymptotic forms of problem (2.2) have the form $F'(\eta)=\sqrt{\pi/\beta}\,\mathrm{erfc}\,(^1/_2\eta\sqrt{|\beta|})$. We note that, for all $|\beta|\geqslant10$, the asymptotic values of $F'(0)$ differ from the numerical values by less than 2%. In the case of a fixed β and increasing η, the function $F'(\eta)$ decays monotonically and $F(\eta)$ increases monotonically and tends to a finite limit. When $\beta=-1$, we present the numerical values $F'(0)=1.3115$ and $F(\infty)=1.1410$. The thickness of the boundary-layer decreases monotonically as $|\beta|$ increases.

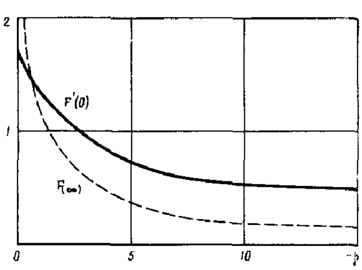

Fig.1

When $\beta>0$, the selfsimilar solution is constructed in the form $\psi=\sqrt{\varphi}\tau^{1/3}F_1$ with the condition at infinity $F_1'(\infty)=0$. The function $F_1(\eta)$ is obtained using the formula $F_1=F+\alpha\eta$, where $\alpha\geqslant\beta$ and $F(\eta)$ satisfies problem (2.2) in which β has to be replaced by $-|\beta|$.

Let us now determine the leading term in the asymptotic expansion (1.3) for a pressure in the boundary-layer q_0. By applying a second iterative process /9/ to system (1.1), constructed on the normal to the free boundary, we derive an equation for q_0 from which it follows that

$$q_0=-k\int\limits_s^\infty h_{\varphi0}^2\,ds \qquad (2.3)$$

Let us now find the value of q_0 on the free boundary. By integrating the first equation in (2.1) with respect to s on the semi-axis $[0,\infty)$, using integration by parts and integrating the resulting expression with respect to φ, we derive the relationship

$$\int\limits_0^\infty h_{\varphi0}^2\,ds-V\int\limits_0^\infty h_{\varphi0}\,ds=\int\limits_{\varphi_0}^{\varphi}T\,d\varphi+\int\limits_0^\infty f_0(f_0-V)\,ds \qquad (2.4)$$

Here, $f_0=h_{\varphi0}(s,\varphi_0)$ is the velocity profile in the boundary-layer in the section $\varphi=\varphi_0$. By putting $s=0$ in (2.3) and taking account of (2.4), we will find the value of q_0 on the free boundary

$$q_0|_\Gamma = k\left[\int_{\varphi_*}^{\varphi} T\,d\varphi + V\int_0^\infty h_{\varphi0}\,ds + \int_0^\infty f_0(f_0 - V)\,ds\right] \tag{2.5}$$

We note that, when $V = 0$, the value of $q_{0\Gamma}$ can be determined without the solution of the boundary layer problem (2.1) if the velocity profile in the boundary layer is known in a certain cross-section $\varphi = \varphi_0$ (in the neighbourhood of the critical point, for example, /7/).

The functions $\mathbf{v}_1$, p_0 and ξ_0 which determine the non-viscous flow outside of the boundary layer and the asymptotic form of the free boundary are obtained by application of the first iterative process /9/ to system (1.1).

We denote by Γ_0, the free boundary $z = \xi_0$ of the non-viscous flow. Close to the surface Γ_0, we introduce the local orthogonal coordinates ξ_1, φ_1, where ξ_1 is the distance to the boundary Γ_0. We represent the curvature of the curve Γ in the form $k = k_0 + \varepsilon^{1/2}k_1 + \ldots$, where k_0 is the curvature of the curve Γ_0. Upon substituting the expansions (1.2) into system (1.1), allowing for the fact that $\mathbf{h}_0 = \mathbf{h}_1 = \mathbf{w}_1 = q_0 = q_1 = 0$ outside of the boundary-layer and equating the coefficients of ε^0 and ε to zero, we obtain a boundary-value problem for $\mathbf{v}_1$, p_0 and ξ_0 which we present in the dimensional form

$$\partial\mathbf{v}_1/\partial t_1 + (\mathbf{v}_1, \nabla)\mathbf{v}_1 = -\rho^{-1}\nabla p_0, \quad \operatorname{div}\mathbf{v}_1 = 0 \tag{2.6}$$

$$p_0 = \rho g z - k_0\left[\sigma + \int_{\varphi_*}^{\varphi} T\,d\varphi + c\int_0^\infty h_{\varphi0}\,ds + \int_0^\infty f_0(f_0 - c)\,ds\right], \quad (x, z) \in \Gamma_0$$

$$\partial\xi_0 / \partial t + v_{x_1}\partial\xi_0 / \partial x_1 = v_{z1}, \quad (x, z) \in \Gamma_0; \ \mathbf{v}_1\mathbf{n}_1|_S = 0$$

Here, account has been taken of the fact that $h_{z1} = 0$ and that $\mathbf{n}_1$ is the normal to the solid boundary S.

Hence, the action of tangential stresses on the free boundary of a low viscosity fluid leads to the appearance, in the dynamic boundary condition on the free boundary of the flow of an ideal fluid (2.6), of additional terms which depend on the curvature of the boundary, the velocity field in the boundary layer and the tangential load.

The vector-function $\mathbf{w}_1$ determines the velocity field in the boundary layer close to the solid wall S and compensates for the discrepancy which arises in complying with the sticking condition on S with the vector $\mathbf{v}_1$. The boundary-value problem for $\mathbf{w}_1, r_1$ is not given since these functions make a contribution to the elevation of the free boundary in the higher approximations starting from the second.

3. Let us now consider the problem of the effect of surface tangential stresses on the long waves which are propagating at a velocity c in a layer of thickness H. By assuming that the flow of an ideal fluid, which satisfies (2.6) is vortex-free, we define the velocity potential Φ by the relationship $\mathbf{v}_1 = \nabla\Phi$.

The parameter $\delta = H/L$ is introduced, where L is the characteristic horizontal scale of the flow (the wave length, for example). Then, $\delta \ll 1$ in the case of long waves. The condition for the thickness of the boundary layer to be small compared with the dimensionless thickness of the layer δ leads to the relationship $\rho\nu^2 LH^{-3}T_*^{-1} \ll 1$. In system (2.6), we change to the dimensionless variables $t' = \delta t\sqrt{g/H}$, $z' = z/H$, $x' = \delta x/H$, $\Phi' = \delta H\sqrt{gH}\Phi$ and introduce the moving coordinate $x_1 = x' - t'$, $z_1 = z'$ and the slow time $t_1 = \delta^2 t'$. The tangential stress is represented in the form $T = \delta T_1$.

The equations of the long waves are obtained using the method in /10/ which is based on an expansion of the solution in series in powers of a small parameter

$$\Phi = \Phi_0 + \delta^2\Phi_1 + \ldots, \quad \xi_0 = 1 + \delta^2\xi_{01} + \ldots$$

The function $u = \xi_{01}$, which determines the main correction to the elevation of the free boundary, satisfies the non-linear equation

$$\frac{\partial u}{\partial t_1} + \frac{3}{2}u\frac{\partial u}{\partial x_1} + \frac{1}{2}\frac{\partial}{\partial x_1}\left[\left(\frac{1}{3} - \sigma_0 - \int_0^{x_1} T_1\,dx_1 - V\int_0^\infty h_{\varphi0}\,ds + \int_0^\infty f_0(V - f_0)\,ds\right)\frac{\partial^2 u}{\partial x_1^2}\right] = 0 \tag{3.1}$$

Here, $\sigma_0 = \sigma T_*^{-1}L^{-1}$ is the dimensionless surface tension and $f_0(s)$ is the longitudinal component of the velocity in the boundary layer in the section $x_1 = 0$. Eq.(3.1) generalizes the Korteweg-de Vries equation in the theory of long non-linear waves /10/ to the case of the action of surface tangential stresses.

We now represent the velocity of propagation of the waves in the form of a series: $c/\sqrt{gH} = 1 + \delta^2 c_1 + \ldots$. We note that, when $T_1 = 0$, solitary waves are contained among the solutions of Eqs.(3.1), which move at a supercritical velocity when $c_1 > 0$, and periodic waves in the case when $c_1 < 0$ /10/.

Let us now determine the solutions of the travelling wave type $u(x - c_1 t)$ in the case of the asymmetric tangential load $T_1 = 2\lambda \exp[-(\theta - 1)^2]/3$ where $\theta = x_1 - c_1 t$ in the case when $V \ll 1$. It can be shown that $f_0 = h_{\varphi 0}|_{\theta=0} = 0$ in Eq.(3.1). In order to do this close to $\theta = 0$, by approximating the load by a linear function we find the solution of the boundary layer Eqs.(2.1), where $h_{\varphi 0} = a^2 \theta \exp(-as)$, $a = [2\lambda \exp(-1)/3]^{1/3}$.

Now, by neglecting the surface tension in (3.1) and putting $V = 0$, we derive the equation

$$\left(1 - 3\int_0^\theta T_1\, d\theta\right) u'' + 4{,}5 u^2 - 6 c_1 u = 0 \tag{3.2}$$

The free boundary which is obtained as the result of the numerical solution of Eq.(3.2) for various values of the amplitude of λ when $c_1 = 0.5$ is shown in Fig.2. When $\lambda = 0$, Eq.(3.2) describe a soliton (curve *1*) /10/. For values of λ in the interval $0 \leqslant \lambda \leqslant \lambda_1 = 0.2751$, the "peak" of the soliton is displaced to the left. When $\lambda_1 < \lambda < \lambda_2 = 3.624$, waves appear from the right-hand side of the peak, the amplitudes of which increase as λ increases (the typical form of a wave is shown in curve *2*). When $\lambda = \lambda_2$, a wave of the "hump" type is obtained (curve *3*) for which $u \to 0$ and $\theta \to -\infty$ and $u \to {}^2/_3$ when $\theta \to \infty$. No bounded numerical solutions are found for values of $\lambda > \lambda_2$. The results of numerical calculations of the form of the wave which travels with a subcritical velocity when $c_1 = -0.5$ and $\lambda = 0.25$ are shown in Fig.3. The tangential load leads to a decrease in the amplitude of the wave and its length. This effect is less significant, the greater the amplitude of the load.

Numerical calculations were also carried out into the action of a symmetric tangential load $T_1 = {}^2/_3 \lambda \exp(-\theta^2)$ on the free boundary for $c_1 = 0.5$. In this case, when $0 \leqslant \lambda \leqslant 1$, a family of solitons is obtained for which the height of the hump increases as λ increases and attains a value of 1.184 when $\lambda = 1$. In the interval $1 \leqslant \lambda \leqslant \lambda_* = 1.667$, waves appear on both sides of the hump while the lengths and amplitudes of the waves increase as λ becomes larger. When $\lambda = \lambda_*$, the solution is a soliton for which $u(0) = 1.50$ and $u(\pm\infty) = {}^2/_3$. There were no numerical solutions for $\lambda > \lambda_*$. Typical plots of a wave form are depicted in Fig.4. Curves *1-4* correspond to values of λ equal to 0, 1, 1.2 and λ_*.

Solutions of the travelling-wave type were studied numerically in the case of Eq.(3.1) when $V = -1$ and $T_1 = \lambda\sqrt{\theta}/\theta > 0$), $T_1 = 0$ $(\theta < 0)$. In this case, the velocity in the boundary layer, $h_{\varphi 0} = 0$ $(\theta < 0)$, $h_{\varphi 0} = F'(\eta)$ $(\theta > 0)$, where the function $F(\eta)$ is determined from the boundary-value problem (2.2). Solitary waves are obtained when $\lambda > 0$ for which the hump is displaced to the right from the line $\theta = 0$ and the height of the hump is only slightly greater than unity.

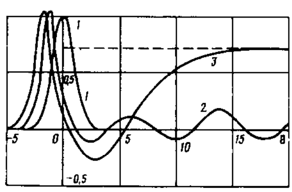

Fig.2

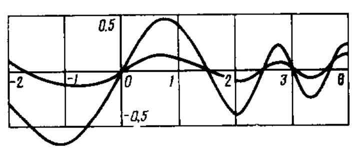

Fig.3

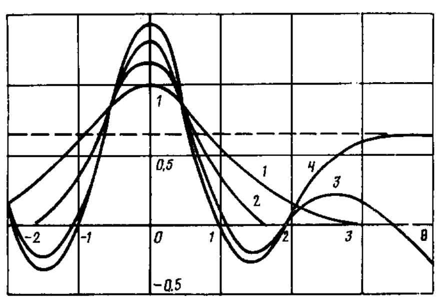

Fig.4

4. Let us investigate the effect of tangential stresses on the propagation of small amplitude progressive capillary-gravitational waves. We assume that a system of tangential stresses $T(x-ct)$ shifts over a flow surface which is moving at a velocity w. By assuming that the flow of a non-viscous flow is slow and potential, by linearizing system (2.6) we obtain a boundary-value problem in the stream function $\psi(\varphi, z)$ $(\varphi = x - ct)$ which determines the travelling waves

$$\begin{gathered}\Delta\psi = 0 \\ (c-w)^2\frac{\partial\psi}{\partial z} - g\psi = -\rho^{-1}f(\varphi)\frac{\partial^2\psi}{\partial z^2} \quad (z=0) \\ \partial\psi/\partial\varphi = 0 \quad (z=-H) \\ f(\varphi) = \sigma + \int_0^{\varphi} T\,d\varphi + c\int_0^{\infty} h_{\varphi 0}\,ds + \int_0^{\infty} f_0(f_0 - c)\,ds\end{gathered} \tag{4.1}$$

Let us suppose that the tangential stresses are specified locally by means of a delta-function $T = \tau\delta(\varphi)$, where $\tau \geqslant 0$ and the characteristic velocity in the boundary layer is substantially greater than the velocity $c\,(U \gg c)$. Then, apart from small high-order quantities

$$f(\varphi) = \sigma + \tau \ (\varphi > 0), \ f(\varphi) = \sigma \ (\varphi < 0)$$

The solution of problem (4.1) is obtained in the form

$$\psi_i = A_i \operatorname{sh}\Omega_i(z+H)\sin\Omega_i\varphi$$

where $i=1$ when $\varphi > 0$ and $i=2$ when $\varphi < 0$. The wavenumber Ω_1 satisfies the dispersion equation

$$F_1(\Omega_1) \equiv (\tau+\sigma)\,\Omega_1^2 \operatorname{th}\Omega_1 H - \rho\Omega_1(w-c)^2 + \rho g \operatorname{th}\Omega_1 H = 0$$

The parameter Ω_2 is the solution of this equation when $\tau = 0$. An investigation of the roots of the dispersion equation shows that, when $(w-c)^2 < gH$, two solutions exist which correspond to gravitational and capillary waves. These solutions are bounded when $\tau < \tau_*$ and become unbounded when $\tau \geqslant \tau_*$. The value of τ_* was determined numerically from the equation which is obtained when Ω_1 is eliminated from the system $F_1 = 0$, $\partial F_1/\partial\Omega_1 = 0$. We note that $\tau_* = 0$ when $w = c$ and, as the parameter $\gamma = (w-c)^2/(gH)$ becomes larger, the value of τ_* increases monotonically and, when $\gamma = 1$, it reaches a maximum value equal to $\tau_m = {}^1/_3\rho g H^3 - \sigma_0$.

Analysis of the amplitudes $A_i(\tau, \sigma, g, w-c, H)$ shows that the tangential stresses suppress gravitational waves and reinforce capillary waves. When $(w-c)^2 = gH$, $\tau < \tau_m$, only capillary waves exist and the amplitudes of these become unbounded when $\tau \geqslant \tau_m$. If, however, $(w-c)^2 > gH$, then, for any $\tau > 0$, there exist only capillary waves and the amplitudes and wavelengths of these waves increase as τ becomes larger.

In the case of a layer of infinite depth, gravitational and capillary waves only exist when $\tau < \tau_0 = -\sigma + {}^1/_4\rho(w-c)^4 g^{-1}$. The waves decay (their amplitudes become unbounded as $\varphi \to \infty$) when the tangential load attains the critical value τ_0.

Let tangential stresses of the form $T = \lambda f(\varphi)$ $(\varphi > 0)$, $T = 0$ $(\varphi < 0)$, where $f(\varphi)$ is a function which is integrable on the half-axis, be specified on the free boundary. It can be shown that the critical value of the tangential load at which the amplitudes of the waves become unbounded does not exceed the value

$$\lambda_* = [{}^1/_4\rho(w-c)^4 g^{-1} - \sigma] \Big/ \int_0^{\infty} f(\varphi)\,d\varphi$$

As in a layer of finite depth, the action of the tangential stresses leads to the decay of the graviational waves and to reinforcement of the capillary waves.

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THE GENERALIZED PROBLEM OF BREAKUP OF AN ARBITRARY DISCONTINUITY*

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The problem of breakup of an arbitrary discontinuity in a gas (the Riemann problem) is generalized to the case when an arbitrary, in general space-variable, distribution of the gas-dynamic parameters is given on both sides of the discontinuity at the initial instant of time (the generalized Riemann problem /1/). The solvability of this, in general non-selfsimilar, model is proved and analytical formulas are found for its solution in a small neighbourhood of the points of discontinuity in the x, t plane, where x is the space coordinate and t is the time.

A complete analysis of the selfsimilar Riemann problem was developed by Kochin /2/. The generalized Riemann problem is in general non-selfsimilar and does not admit of a simple analytical solution over the entire x, t plane. However, some analytical solutions may be obtained for this problem. Thus, for a linear initial distribution, analytical formulas were obtained in /1/ for the values of the derivatives of the gas-dynamic parameters along the contact discontinuity for $t=0$.

Below, the generalized Riemann problem is considered in a small neighbourhood of the point of discontinuity in the (x, t) plane and its analytical solution is constructed to a first approximation in $\theta=\sqrt{x^2+t^2}$. Analytical formulas for the trajectories of discontinuities are obtained in the same approximation.

1. The generalized Riemann problem is reducible to the following Cauchy problem for one-dimensional non-stationary equations of gas dynamics:

$$(\rho\varphi)_t+(\rho u\varphi+\mathbf{F})_x=0 \tag{1.1}$$
$$\varphi=(1,\ u,\ e+{}^1/{}_2u^2)^T,$$
$$\mathbf{F}=(0,p,pu)^T,\quad \rho\varphi(0,x)=\begin{cases}\varphi_1(x), & x<0\\ \varphi_2(x), & x>0\end{cases}$$

where u, ρ and e are the velocity, density, and the specific internal energy, $p=p(\rho,\ e)$ is the pressure and $\varphi_1(x)$ and $\varphi_2(x)$ are functions which are differentiable in the domain of definition, which specify the initial parameter distribution.

We will rewrite the system of Eqs.(1.1) in characteristic form, introducing the specific

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